A naturalistic justification of the generic multiverse with a core

Matteo de Ceglie

Salzburg University, Salzburg, Österreich
deceglimatteo@gmail.com

The importance of set theory can hardly be overestimated: from its first development by Cantor and Zermelo, to the most recent results, set theory has always been seen as providing a foundation of all of mathematics. And this is for an important reason: we can actually model and develop all mathematics in set theory. That is, we can use the language of set theory to formalize the whole of mathematics. However, while set-theory is usually identified with its standard axiomatisation, ZFC, it is highly unclear whether such an axiomatisation adequately captures our conception of sets, and indeed what exactly such a conception is. The seminal work of Zermelo first suggested a 'cumulative' conception of set theory, according to which sets are constructed in stages and form a v-shaped structure, the universe of set theory V. However, from the sixties onwards it became clear that some propositions in the language of set theory cannot be proved from the canonical axioms of ZFC — the Continuum Hypothesis (CH), a central set-theoretic claim, being a case point. Together with Gödel's First Incompleteness Theorem, according to which systems much weaker than ZFC cannot decide all the sentences of their language, if consistent, the independence of CH from ZFC strongly suggested that the usual set theory was inadequate as it stood. Whence the quest for finding an adequate extension of ZFC started: new axioms were proposed, but none of them was stably accepted as part of set theory. The reason is that several such new axioms are mutually exclusive: choosing one implies that all the results proved under an incompatible axiom can no longer be proven. This led to competing "set theories", and to the necessity to develop a formal methodology to compare the axiom candidates. At present, none of these theories is accepted as the "new" set theory.

One of the most recent development considers set theory (and hence mathematics) not as a unique, single universe, but as a *multiverse*.[1] That is, the thought goes, all the competing theories are all description of various universes inside the multiverse, and set theory, now taken in a broader sense, is the study of such a multiverse.

Our understanding of the multiverse is still very partial, however. While several philosophical considerations militate in favour of a multiverse approach (for instance, it is sometimes advocated on the grounds that it allows a more precise characterization of independence results), it is not fully clear what the multiverse actually is, since its very nature changes together with the various philosophical assumptions that are made to justify its very existence. From a mathematical point of view, the possible multiverses can be classified according to their hierarchical character, or lack thereof. At one extreme we have the *broad* multiverse, with no hierarchy at all; at the other the *vertical* multiverse, with a very strong hierarchy[2]. From a more philosophical point of view, one can also consider their ontological commitments. A realist multiverse is committed to the existence of its own universe, an anti-realist one isn't. Therefor, we can have a *realist* and an *anti-realist* broad multiverse, or a realist and an anti-realist vertical multiverse.[3] Thus, rejecting the standard cumulative conception of set-theory, or any

[1]See, for a general account, [1].
[2]For a detailed account of the broad multiverse refer to [11].
[3]For some details on this classification see [15].

conception of sets according to which there is a single universe of sets, by no means settles the question concerning the nature and justification of the multiverse.

My project is devoted to the justification of one particular type of multiverse, the *generic multiverse with a core* (GM_H from now on), developed by John R. Steel (in [23]), from a naturalist point of view. Naturalism in philosophy of mathematics is a methodological approach, which recommends considering mathematical practice as the ultimate arbiter of all questions regarding the philosophy of mathematics. For example, if mathematicians say that natural numbers exist, then natural number exist; if mathematicians use the Axiom of Choice, then any set theory worth its salt should contain it; and so on. In our case, this means justifying the GM_H with particular attention to mathematical practice: if it turns out to be better for mathematical practice, then it should be adopted.

In order to meet this goal we need, first of all, a mathematical characterization of the GM_H. Intuitively, the GM_H lies in a middle-ground between the broad multiverse and the vertical multiverse: it does not feature a strong hierarchy as the vertical multiverse, but it provides a criterion to choose between universes (by contrast, the broad multiverse provides no criterion at all). This is the existence of a *core*: a set of truths that is common to all the universes. Only after having fully developed this mathematical framework can we be in a position to philosophically assess it, and to make progress in the quest for discovering the true nature of sets.

My project is therefore interdisciplinary in its core: the philosophical goal is based on mathematical results and arguments. From a philosophical point of view this means integrating *naturalism* and *pluralism* in philosophy of mathematics. More specifically, my goal is to justify a pluralist conception of set theory (namely, the multiverse GM_H) from a naturalist point of view, using the main naturalistic principles developed by Penelope Maddy[4]: MAXIMIZE and UNIFY. MAXIMIZE asserts that we should prefer a theory that allows us to prove more isomorphisms. According to UNIFY, the theory we are analyzing should be foundational. Both principles are naturalistically desirable: MAXIMIZE provides us a theory that is powerful enough to be a Generous Arena where studying and analyzing all mathematical objects and methods, while UNIFY assures us that this theory is still a possible foundation for mathematics. But since these two principles must be applied to a precise mathematical structure, we first need to characterize and analyze the GM_H with a core from a mathematical point of view. To do so we will exploit Axiom H in a way that forces the multiverse to be as we need it to be. More specifically, Axiom H implies the existence of the multiverse's core, and hence gives the multiverse a very precise shape: from a common core, composed by a minimal set theory common to all the universes, we get branching alternative universes, that extends this core to every direction. I will argue that such a characterization satisfies both MAXIMIZE and UNIFY, and that is indeed does so – pace Maddy – to a better degree than the standardly accepted ZFC. In turn, this strongly suggests that that the GM_H provides the most adequate framework for mathematical practice: a framework that is foundational — that allows to model all present mathematics — and that maximizes the number of possible isomorphisms.

This work is needed and novel since, at present, no consensus has been reached, among philosophers and set-theorists, on the question which set-theoretic universe (and theory) is better suited for mathematical practice. Some such universes have been extensively studied, but they all still have the same status: competing theories without any real advantage over the others. This project is mainly devoted to settling this debate, arguing that the multiverse conception, as captured by the GM_H, is the best one both for set theory and mathematical practice.

[4]For a detailed description of these two principles, see [16] and [17].

References

[1] Caroline Antos, Sy-David Friedman, Radek Honzik, and Claudio Ternullo. Multiverse conceptions in set theory. (192):2463–2488, 2015.

[2] Tatiana Arrigoni and Sy-David Friedman. The hyperuniverse program. (19):77–96, 2013.

[3] Mark Balaguer. A platonist epistemology. (103):303–325, 1995.

[4] Mark Balaguer. *Platonism and Anti-Platonism in Mathematics*. Oxford University Press, 1998.

[5] Solomon Feferman. Does mathematics need new axioms? (106):99–111, 1999.

[6] Solomon Feferman. Why the program for new axioms need to be questioned. (6):401–446, 2000.

[7] Solomon Feferman. Conceptions of the continuum. (51):1–19, 2009.

[8] Solomon Feferman. Is the continuum hypothesis a definite mathematical problem? testo di una lezione all'interno dell'EFI Project, 2011.

[9] Chris Freiling. Axioms of symmetry: throwing darts at the real line. (51):190–200, 1986.

[10] Victoria Gitman and Joel David Hamkins. A natural model of the multiverse axioms. (51):475–484, 2010.

[11] Joel David Hamkins. The set-theoretic multiverse. (5):416–449, 2012.

[12] Leon Horsten. Platonic formalism. (54):173 – 194, 2001.

[13] Luca Incurvati. Too naturalistic and not naturalistic enough: Reply to horsten. (69):261 – 274, 2008.

[14] Peter Koellner. Hamkins on the multiverse. 2003.

[15] Peter Koellner. *Truth in Mathematics: the Question of Pluralism*. Palgrave Macmillan, 2009.

[16] Penelope Maddy. *Naturalism in Mathematics*. Oxford University Press, 1997.

[17] Penelope Maddy. *Defending the Axioms*. Oxford University Press, 2011.

[18] Donald Martin. Multiple universes of sets and indeterminate truth values. (20):5–16, 2001.

[19] Stewart Shapiro. *Thinking about Mathematics*. Oxford University Press, 2000.

[20] Saharon Shelah. The generalized continuum hypothesis revisited. (116):285–321, 2000.

[21] Saharon Shelah. The future of set theory. In Haim Judah, editor, *Israel Mathematical Conference Proceedings*, 2002.

[22] Saharon Shelah. Logical dreams. (40):203–228, 2003.

[23] John R. Steel. *Gödel's program*. Cambridge University Press, 2014.

[24] Hao Wang. *A Logical Journey*. MIT Press, 1996.

[25] William Hugh Woodin. The continuum hypothesis i. (48):567 – 576, 2001.

[26] William Hugh Woodin. The continuum hypothesis ii. (48):681 – 690, 2001.

[27] William Hugh Woodin. The continuum hypothesis. In Rene Cori, Alexander Razborov, Stevo Todorĉević, and Carol Wood, editors, *Logic Colloquium 2000*, 2005.

[28] William Hugh Woodin. *The Continuum Hypothesis, the Generic-Multiverse of Sets and the* Ω *Conjecture*. Cambridge University Press, 2011.